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The salt-finger zone

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In order to investigate the stability of infinitely long fully developed salt fingers Stern (1975) has proposed a model in which the basic configuration is independent of the vertical and is sinusoidal in the horizontal direction, with constant background gradients of temperature and salinity. The present study deals with a model of finite vertical extent where τ, the ratio of the diffusivities of salt and heat, is small, and where the constant background salt gradient is replaced by a salt difference between the reservoirs above and below a salt-finger region of finite depth. Steady-state solutions in two and three dimensions are obtained for the zero-order ($\tau = 0$) state in which rising (sinking) fingers have the salinity of the lower (upper) reservoir. For two-dimensional fingers the horizontal scale corresponding to maximum buoyancy flux turns out to be 1.7 times the buoyancy-layer scale associated with the background stable temperature gradient. Heat, salt and buoyancy fluxes are calculated. A boundary-layer analysis is given for the (salt) diffusive correction to the zero-order solution. The same set of calculations is carried out for salt fingers in a Hele-Shaw cell. An assessment of Schmitt's (1979*a*) model of a finger zone of finite depth shows that the parametric restrictions required by the model cannot be satisfied when Stern's idealization is used for the final state. The present model appears to be preferable for constructing a Schmitt-like theory for $\tau \ll 1$.

1. Introduction

In double-diffusive convection with the less diffusible substance destabilizing and the more diffusible one stabilizing – e.g. hot and salty above, cold and fresh below – one generally obtains a series of layers of nearly homogeneous fluid, separated by zones occupied by salt fingers. A model for such developed finger zones was given by Stern (1975), based on a solution of the equations that is periodic in the horizontal (x) and independent of the vertical (z), except for 'background' uniform gradients of salt (s) and temperature (T). The relevant equations are

$$w_t + p_z - g(\alpha T - \beta s) = \nu w_{xx}, \tag{1a}$$

$$T_t + w\bar{T}_z = \kappa_T T_{xx}, \tag{1b}$$

$$s_t + w\bar{S}_z = \kappa_S s_{xx}. \tag{1c}$$

Here $\bar{T}_z$ and $\bar{S}_z$ are constants, and T and s represent z-independent deviations from these background constant gradients. The vertical velocity w is also independent of z, and the horizontal velocity components are zero. These equations have solutions in which $p = 0$ and the other variables are multiples (denoted by the same letters) of $\sin(kx)\exp(\lambda t)$; these multiples satisfy the equations

$$\lambda w - g(\alpha T - \beta s) = -k^2 \nu w, \tag{2a}$$

$$\lambda T + w\overline{T}_z = -k^2 \kappa_T T, \tag{2b}$$

$$\lambda s + w\overline{S}_z = -k^2 \kappa_S s, \tag{2c}$$

which may be regarded as determining λ (through the condition of existence of a non-zero solution), or as fixing a relation among the other parameters if λ is given.

If we introduce the parameters $\sigma = \nu/\kappa_T$ and $\tau = \kappa_S/\kappa_T$, and set $\lambda = \kappa_T k^2 \Lambda$, $R_T = \alpha g \overline{T}_z/(k^4 \nu \kappa_T)$, $R_S = \beta g \overline{S}_z/(k^4 \nu \kappa_T)$, the condition for existence of a non-zero solution is

$$(\Lambda+1)(\Lambda+\tau)\left(\frac{\Lambda}{\sigma}+1\right) - (\Lambda+1)R_S + (\Lambda+\tau)R_T = 0. \tag{3a}$$

In particular there is a steady solution ($\Lambda = 0$) if

$$R_S = \tau(R_T+1). \tag{3b}$$

This condition may be used to calculate k if the gradients are specified, and actually gives a reasonable estimate of the horizontal size of salt fingers. It should be noted that although these calculations involve only very simple linear equations, this is not a linearized model but an exact solution of the Boussinesq equations. However, it has the somewhat disturbing feature that if k is smaller than the value for a steady solution just mentioned, then there is also an exact solution, and it grows exponentially in time. Furthermore, if attention is – rather arbitrarily – restricted to the steady case, there is not just one solution: the overall amplitude remains entirely arbitrary. Various not altogether conclusive discussions referring to stability considerations, etc. have been put forward to try to relate this attractive and simple solution more closely to real salt-finger zones.

Stern's solution, or some modification of it, may be appropriate in the middle of the salt-finger zone. However, since the height of the finger zone is observed to be limited to a scale much less than the depth of the fluid, a more realistic dynamical model must include dependence on the vertical coordinate. This realization induced Stern (1969) to look into the stability of an array of infinitely long fingers. His conclusion, subsequently confirmed by a more rigorous analysis by Holyer (1984), is that a collective instability can occur, i.e. that the system can be unstable to a disturbance with a vertical wavelength much larger than the finger width when D, the ratio of the buoyancy flux to $\nu(\alpha T_z - \beta S_z)$, is of order unity. In a later paper Stern (1976) assumed a slightly supercritical value of D, together with statistical assumptions about the isotropy of a quasi-laminar regime of salt fingers and the assumption that the buoyancy flux is maximum, to obtain the result that the ratio of convective heat to salt flux is 1/4. Concomitant parts of the theory are that the mean salt gradient in the finger zone vanishes and that the depth of the fingers is finite.

An altogether different approach to the finite-depth problem was adopted by Schmitt (1979*a*), who put aside any consideration of collective instability and focused on the time-dependent solutions of Stern's idealized model for infinitely long fingers. Schmitt argued that fingers would be generated in the transition regions and penetrate into the finger zone where they have the characteristic features of fingers with maximum growth rate. He then assumed that the wavelength of the fingers remains constant, that the mean temperature gradient is unaltered, and that the fingers evolve to the equilibrium state described by (3*b*). A necessary part of that evolution is a decrease in the mean salinity gradient from the value associated with the basic state. The reduced gradient can be calculated from the initial gradients, the wavelength of maximum growth rate, and the observed depth of the finger zone. The theoretically deduced flux ratios agree rather well with the values measured in experiments.

In addressing the problem of fingers of finite depth both Stern and Schmitt have started with the infinitely deep, constant-gradient model. To this they add (in their different ways) other considerations which serve to tie down the undetermined quantities in Stern's idealization. We shall present here a different idealized model, which can play a role analogous to that of Stern's in more extended theoretical studies. Our idealization is directed to the finite-depth case (though this fact is somewhat concealed in the zero-order model) and we believe it usually gives a description that is closer to the physics of real salt-finger flows than Stern's idealization does. Our analysis is based on two observed features of salt fingers. The first is that salt fingers that evolve from an initial configuration of two layers are driven by a destabilizing salt difference between the two reservoirs rather than a mean gradient of salt. Therefore, only a finite amount of potential energy is available (whereas a uniform salinity gradient in a vertically infinite system has infinite potential energy). The second is that the fluid ejected into a reservoir by a finger often has nearly the salinity anomaly of the other reservoir, i.e. salt diffusion is incapable of smoothing out the salinity anomaly in the time that it takes the fluid to cross the salt-finger zone. The latter phenomenon, observed in both experiments and the numerical calculation of Piascek & Toomre (1980), is due to a combination of slow diffusion of salt and rapid fluid flow in the finger. It is observed whenever the fingers are not too long and the salt difference is large.†

The analysis presented below treats fingers of finite length for the case where the salt diffusion coefficient is much smaller than that of temperature. The zero-order model (§ 2) neglects salt diffusion altogether and yields the dependence of temperature and vertical velocity on the horizontal coordinate when the horizontal salinity distribution is specified as a square wave. The first-order model includes the correction to the salinity due to horizontal salt diffusion as the fluid in a finger is carried from one reservoir to the other. In contrast, Stern and Schmitt look for the convective adjustment of a basic state dominated by horizontal salt diffusion. Thus, our starting point is the opposite of theirs. We shall discuss below the circumstances in which one or the other of the idealized models is more appropriate. In Appendix A we show that the assumptions in Schmitt's extension imply restrictions on the density ratio that cannot really be satisfied. We also indicate how the present idealization can be used in more complete models along the lines of Schmitt or Stern. The present system is analysed for a regular fluid and for a Hele-Shaw cell.

† Stern's (1969) original analysis of collective instability is based on a model with $\tau = 0$ and ΔS instead of S_z. The treatment that appears in his book (Stern 1975) assumed $\tau \neq 0$ and S_z instead of ΔS. The two treatments appear to be the same because both solutions are based on an assumed horizontally sinusoidal profile for S. It is evident that in spirit his original model is much closer to ours.

2. The idealized model

2.1. *Viscous fluid*

The exponential growth that can occur in Stern's basic model is made possible by the assumed, infinitely extended, destabilizing salt gradient. Yet fingers are supplied with salt and fresh water from the (homogeneous) reservoirs that bound the finger zone above and below. They are driven by a salt difference rather than a gradient. Stern's model uses gradients because that is the way to obtain a z-independent solution and make the problem tractable. It is, however, true that the fingers are much taller than they are wide, making the idealization $\partial/\partial z \approx 0$ plausible for the perturbed variables. Furthermore, the horizontally averaged temperature is observed to have a nearly constant vertical gradient connecting the reservoir values. These idealizations are retained in the model proposed here. However, particularly when τ is small, the mean salt distribution is more nearly a constant half-way between the reservoir values (Linden 1973); the salinity is not much changed from that of the upper reservoir in the descending regions nor from that of the lower in ascending ones. For these reasons the zero-order form of the model to be discussed below is based on a salt difference and a temperature gradient. Independence of z is also assumed and to make such a model consistent we must avoid the possibility that a mere salt difference will eventually be completely shorted out by horizontal diffusion – thus the zero-order model has $\tau = 0$.

Any model in which $u = v = 0$, $w = w(x, t)$, salinity $= S_0 + z\bar{S}_z + s(x, t)$, temperature $= T_0 + z\bar{T}_z + T(x, t)$ and pressure $= p(x, t)$ (plus a hydrostatic part balancing the x-independent density field) is described by the same equations (1) used above. In the present case we have $\bar{S}_z = 0$, $\kappa_s = 0$, $s = \frac{1}{2}\Delta S$ in descending fingers and $s = -\frac{1}{2}\Delta S$ in ascending ones, so the salinity equation (1c) is trivially satisfied. The quantity ('buoyancy-layer scale')

$$L = \left(\frac{4\nu\kappa_T}{g\alpha\bar{T}_z}\right)^{\frac{1}{4}} \tag{4}$$

has the dimensions of a length; we use it as the basic lengthscale to introduce dimensionless horizontal coordinates x and y. (This choice is justified later in connection with the optimum width given by (11).) We use L^2/κ_T as the time-scale, and set

$$w = \frac{W\kappa_T}{L}, \quad T = \tfrac{1}{2}\theta L\bar{T}_z. \tag{5}$$

The relevant dimensionless equations (with $p = 0$) are

$$\frac{W_t}{\sigma} - 2\theta - W_{xx} - W_{yy} = 2Q, \tag{6a}$$

$$\theta_t + 2W - \theta_{xx} - \theta_{yy} = 0, \tag{6b}$$

where Q is $\beta\Delta S/(\alpha L\bar{T}_z)$ in ascending fingers; in descending fingers the sign of Q is reversed.

We shall consider only periodic arrays of salt-fingers, at first in the two-dimensional case (everything independent of y), and assume that the descending fingers are exactly like the ascending ones except for reversal of the signs of W and θ. Thus we need only consider solutions of (6) on the interval $0 < x < \pi b$, where πbL is the (dimensional) breadth of a finger and b is not yet determined; elsewhere the solution is the odd periodic continuation of that on this interval. Appropriate boundary conditions are $W = \theta = 0$ at $x = 0$, πb. (These make W and θ continuous and continuously differentiable at the finger boundaries.) It is easy to show that all solutions of these equations tend to the steady solution as $t \to \infty$; there is here no question of exponentially growing solutions, and the steady solution is stable to all perturbations that respect the periodicity and z-independence.

The time-independent forms of these equations are the thermal analogues of those of the Ekman layer – in fact, they describe the 'buoyancy layer' (Prandtl 1952), forced by the Q-term which represents the weight of the salinity deviation. The two steady equations can be combined in the single complex equation

$$(W+\mathrm{i}\theta)_{xx} = 2\mathrm{i}(W+\mathrm{i}\theta)-2Q. \tag{7}$$

The solution that satisfies the boundary conditions is readily determined; on $0 \leqslant x \leqslant \pi b$ it is given by the formulae

$$W = Q\frac{\sinh x \sin(\pi b - x) + \sin x \sinh(\pi b - x)}{\cosh \pi b + \cos \pi b}, \tag{8a}$$

$$\theta = Q\left[\frac{\cosh x \cos(\pi b - x) + \cos x \cosh(\pi b - x)}{\cosh \pi b + \cos \pi b} - 1\right]. \tag{8b}$$

If the fingers are not much wider than L, the flow is essentially a row of Poiseuille flows, alternately up and down (figure 1). For wide fingers (large b) this structure becomes a row of antisymmetric buoyancy boundary layers on the interfaces (figure 2) between the regions of different salinity; these boundary layers have the characteristic (dimensional) scale L. It will be noted that for the case displayed in figure 2(*a*) the vertical velocity is much reduced at the centre of the finger. In fact, for any value of $b > 2$, there is a region of downward flow in $0 \leqslant x \leqslant \pi b$. While this is a correct solution of the stated mathematical problem, it is not consistent with the underlying physical picture in which regions of downward flow should always have the salinity of the upper reservoir. Thus the physically relevant range of b is $0 < b \leqslant 2$. If one imagined that b was somehow slowly increased (e.g. by increasing $\overline{T}_z$) in an array of fingers with fixed breadth, then such reversed-flow regions could develop – but they would eventually bring down (or up) fluid of the other salinity, thus effectively splitting apart the 'too wide' finger. In experiments in a Hele-Shaw cell Taylor & Veronis (1985) observed that wide fingers, exposed abruptly to conditions favouring smaller width, become unstable to smaller scale, slanted disturbances which evolve to form a pattern of narrower fingers.

It is clear from the above that the problem also has steady solutions other than the one just given, in which not all fingers are alike. For instance we could have an odd periodic array in which each half-period consisted of two fingers in one direction separated by a thinner finger in the other. (To calculate such solutions one should, of course, use continuity and smoothness at the interfaces rather than $W = \theta = 0$.) There are also many other possibilities, including aperiodic structures. The temptation to investigate such 'spatially chaotic' solutions will, however, be resisted here. Solutions that are periodic in two horizontal dimensions will be discussed briefly

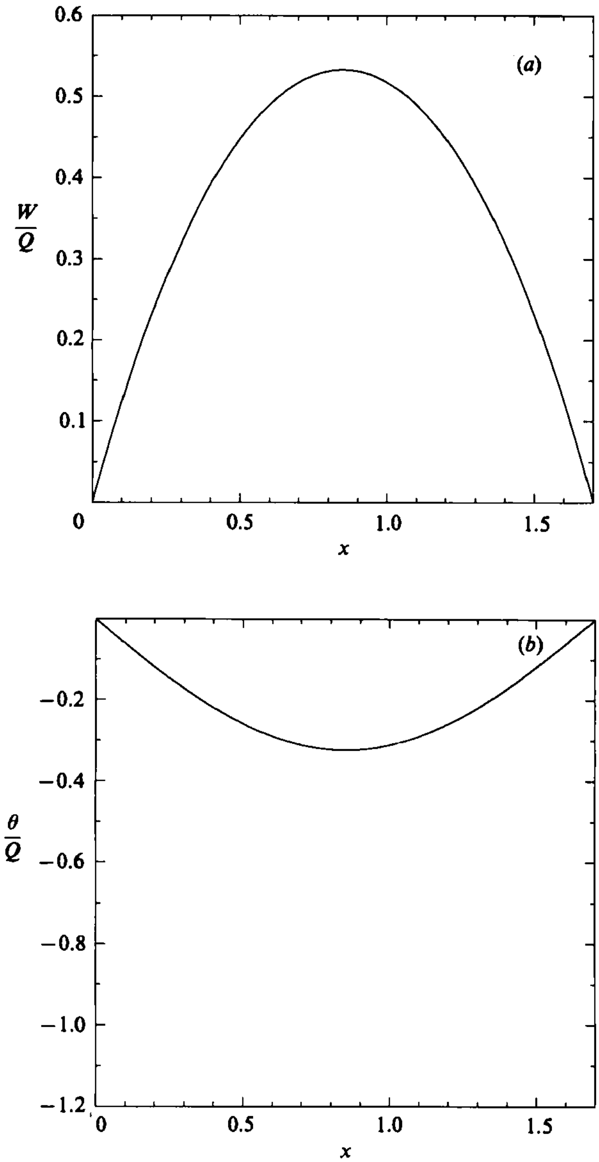

FIGURE 1. (a) W/Q and (b) θ/Q *vs.* x for an ascending finger with a width of L_m $(=1.7L)$, corresponding to maximum buoyancy flux.

below; in a different context D. Loper (private communication) has calculated an analogous axisymmetric solution as a model of an isolated plume.

Horizontally averaged, convective heat (F_T) and salt (F_s) fluxes are defined (in dimensional form) by

$$F_T = \frac{1}{\pi b L}\int_0^{\pi b L} wT\,\mathrm{d}x, \tag{9a}$$

$$F_s = \frac{1}{\pi b L}\int_0^{\pi b L} ws\,\mathrm{d}x, \tag{9b}$$

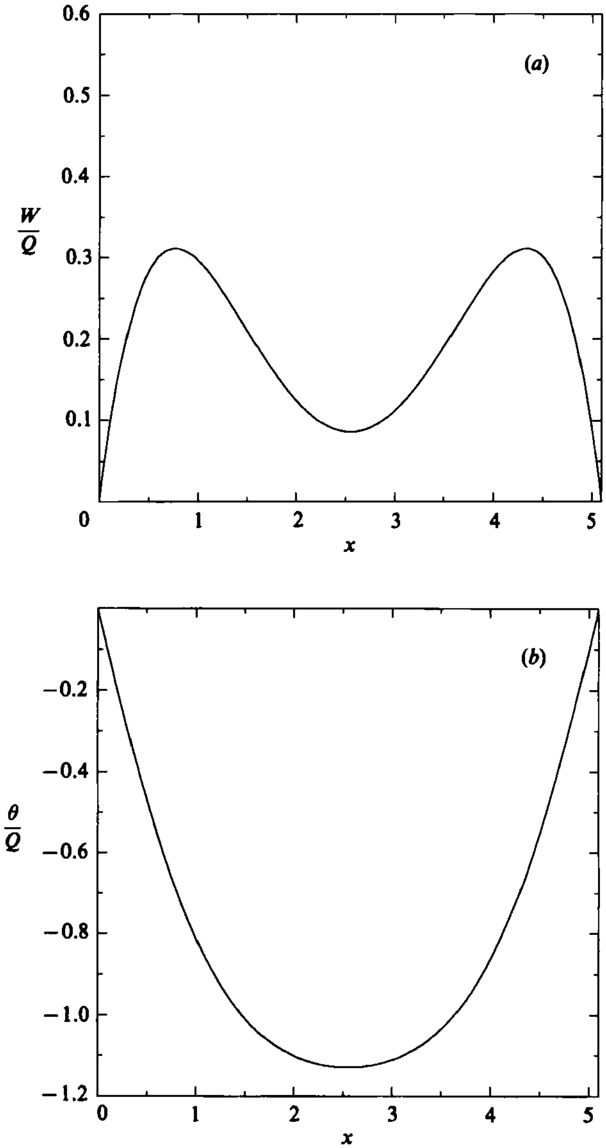

FIGURE 2. (*a*) W/Q and (*b*) θ/Q *vs.* x for ascending finger with a width of $3L_m$. The vertical velocity peaks in the boundary layers and has a lower value at the centre of this (too-wide) finger.

and the buoyancy flux B is $\alpha F_T - \beta F_s$. (The conductive heat flux $-\kappa_T \overline{T}_z$ is omitted here.) These can be shown to be given by the formulae

$$F_T = -\kappa_T \overline{T}_z Q^2 \left(\frac{3}{8\pi b} \frac{\sinh \pi b - \sin \pi b}{\cosh \pi b + \cos \pi b} - \frac{1}{4} \frac{\sin \pi b \sinh \pi b}{(\cosh \pi b + \cos \pi b)^2} \right), \tag{10a}$$

$$F_s = -\frac{\alpha}{\beta} \kappa_T \overline{T}_z Q^2 \left(\frac{1}{2\pi b} \frac{\sinh \pi b - \sin \pi b}{\cosh \pi b + \cos \pi b} \right). \tag{10b}$$

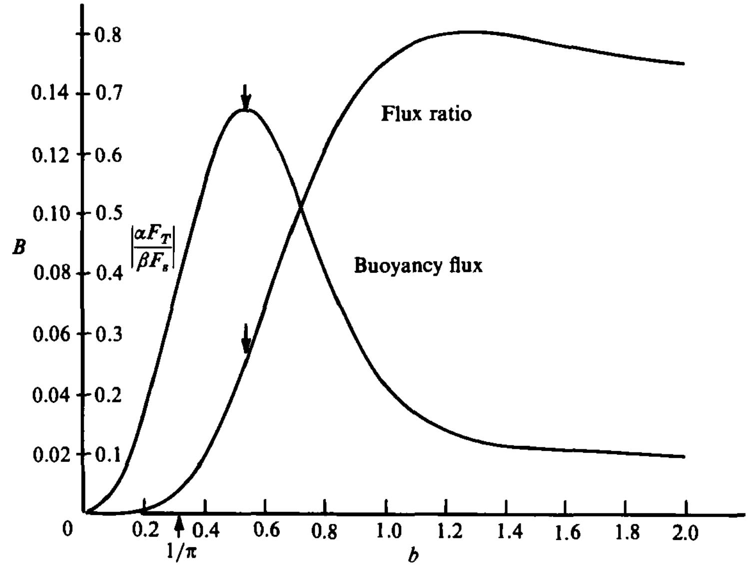

FIGURE 3. Buoyancy flux and flux ratio of heat to salt as functions of b (πb is the width of a two-dimensional finger).

The buoyancy flux as a function of b is shown in figure 3; it has a maximum of $B_m = 0.135\alpha\kappa_T \bar{T}_z Q^2$ at $b_m = 0.542$, which corresponds to a finger width

$$L_m = 1.70\left(\frac{4\nu\kappa_T}{g\alpha\bar{T}_z}\right)^{\frac{1}{4}} \tag{11}$$

or a little under two buoyancy-layer thicknesses.†

The present theory does not in itself specify a value of b appropriate for comparison with experiment (except that it should be < 2). Choosing b_m as the preferred scale has a certain physical and thermodynamic appeal but it has not been justified rigorously. It should be noted that a horizontal scale is not determined by Stern's idealization either; one is obtained only by adding some additional considerations such as those given by Stern (1976) or Schmitt (1979*a*). A detailed discussion of the issue is warranted but in order not to interrupt the flow of the present development, the discussion is relegated to Appendix A. Here, we point out only that Schmitt's mechanistic model implies parametric restrictions that cannot be satisfied and that the present model appears to be a better approximation for the heat–salt system.

Profiles of W and θ for fingers of width L_m are shown in figure 1. The corresponding convective fluxes are given by

$$\alpha F_T = -0.336 B_m, \quad \beta F_s = 1.336 B_m. \tag{12}$$

† When a stably stratified fluid is subjected to thermal forcing at the sides, buoyancy layers provide the transition between the imposed boundary values and the interior fluid. In the present case, rising columns transport cold fluid upward and sinking columns transport warm fluid downward. The buoyancy layers on either side of the finger boundary serve to smooth out these (stabilizing) thermal anomalies, enabling the fluid to release the potential energy of the salinity more efficiently when the width is given by (11) than when the finger is wider. Thinner fingers are too heavily damped by viscosity.

Thus the flux ratio, $\alpha F_T/\beta F_s$, is 0.251, effectively the same as Sterm's value of 0.25 obtained by a totally different argument. This ratio is also shown as a function of b in figure 3. It varies rather rapidly near the width L_m, but is never more than 0.8.

Laboratory measurements of $\alpha F_T/\beta F_s$ across an interface (initially two layers) vary considerably because of the uncertainty about how much heat is lost through the sidewalls during an experiment. For $\alpha\Delta T/\beta\Delta S > 2$ Turner (1967) determined $\alpha F_T/\beta F_s$ to be 0.56; Linden (1971) obtained a value of 0.12 (with heat–sugar experiments); and Schmitt (1979*b*) found that $\alpha F_T/\beta F_s$ decreased from 0.68 for $\alpha\Delta T/\beta\Delta S < 2.5$ to 0.33 for $\alpha\Delta T/\beta\Delta S > 6$.

Salt fingers (really 'sheets') somewhat like those envisioned above have been observed when a shear is imposed across the finger zone, but otherwise fingers are normally three-dimensional with a roughly square planform (cf. Turner 1973, figures 8.18 and 8.19). A solution to the steady form of (6) with a checkerboard pattern of up and down square fingers can be found, though it is not described by formulae as simple as those in the two-dimensional case. Taking a typical ascending finger to be in $0 \leqslant x \leqslant \pi b$, $0 \leqslant y \leqslant \pi b$, one representation of the solution is by a double Fourier series,

$$W + \mathrm{i}\theta = 2Q \sum_{n,m} A_{nm} \sin\frac{nx}{b} \sin\frac{my}{b} \tag{13}$$

in which the coefficients are found to be

$$A_{nm} = -\frac{16}{\pi^2 nm}\frac{b^2}{n^2m^2 + 2\mathrm{i}b^2} \tag{14}$$

if n and m are both odd positive integers, and $A_{nm} = 0$ otherwise. Although this representation is not very suitable for numerical evaluation because of its rather slow convergence, one readily finds from it the following expressions for the convective fluxes;

$$\alpha F_T = -\alpha\kappa_T \bar{T}_z Q^2 \frac{256b^6}{\pi^4} \sum_{\substack{n,m\\ \text{odd}}} \frac{n^2+m^2}{[(n^2+m^2)^2+4b^4]^2 n^2m^2}, \tag{15a}$$

$$-\beta F_s = \alpha\kappa_T \bar{T}_z Q^2 \frac{64b^2}{\pi^4} \sum_{\substack{n,m\\ \text{odd}}} \frac{n^2+m^2}{[(n^2+m^2)^2+4b^4]\, n^2m^2}. \tag{15b}$$

(These series are also somewhat slowly convergent. Numerical evaluation can be facilitated by the following device, illustrated for the case of the series in (15*b*) (all sums are for n, m odd):

$$\sum \frac{n^2+m^2}{[(n^2+m^2)^2+4b^4]\,n^2m^2} = \sum \frac{1}{(n^2+m^2)\,n^2m^2} - 4b^4 \sum \frac{1}{n^2m^2(n^2+m^2)\,[(n^2+m^2)^2+4b^4]}.$$

The second series on the right converges rapidly enough for fairly easy direct evaluation if b is not too large – the difference between the sums up to $n = m = 11$ and those up to 15 are only 10^{-5} even at $b = 3$. The first sum on the right can be transformed to

$$\sum \frac{1}{n^4}\left(\frac{1}{m^2} - \frac{1}{m^2+n^2}\right) = \tfrac{1}{96}\pi^4\,\tfrac{1}{8}\pi^2 - \sum \frac{1}{n^4}\frac{\pi}{4n}\tanh\tfrac{1}{2}\pi n$$

$$= \frac{\pi^6}{768} - \tfrac{1}{4}\pi \sum \frac{1}{n^5} + \tfrac{1}{4}\pi \sum \frac{1}{n^5}(1 - \tanh\tfrac{1}{2}\pi n).$$

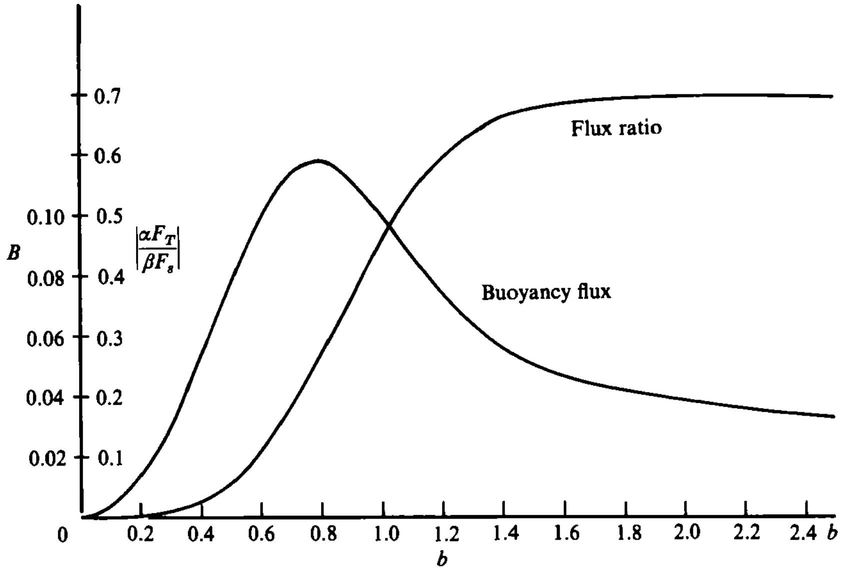

FIGURE 4. Same as figure 3 but for fingers with a square cross-section. Here, πb is the length of one edge of the square finger.

The last sum converges very rapidly (the third term, $n = 5$, is about 10^{-10}) and the other sum is easily seen to be $31\zeta(5)/(32)$ (Riemann zeta function, which is tabulated).)

The buoyancy flux $B = \alpha F_T - \beta F_s$ and the flux ratio $|\alpha F_T/\beta F_s|$ are shown for this case in figure 4. The vertical velocity at the finger centre was also computed (using a similar device to improve convergence) and reverse flow was found to set in when b exceeds 1.485. The maximum buoyancy flux is $0.1174\alpha\kappa_T \overline{T}_z Q^2$, obtained at $b = 0.7844$, which corresponds to a finger edge of about $2.46L$. The flux ratio at this point is 0.256.

2.2. *Fingers in a Hele-Shaw cell*

When the fluid is contained in a Hele-Shaw cell, i.e. a narrow gap between two vertical plates, the conservation equations (1 *b*, *c*) for temperature and salinity are unaltered, but the vertical momentum equation takes the form

$$w_t + p_z - g(\alpha T - \beta s) + \mu w = 0, \tag{16}$$

where $\mu = 12\nu/d^2$ and d is the gap width. The analysis analogous to Stern's once again leads to (3) but with σ, R_T and R_s replaced by $\sigma = \mu/(k^2\kappa_T)$, $R_T = g\alpha\overline{T}_z/(\mu\kappa_T k^2)$ and $R_s = g\beta\overline{S}_z/(\mu\kappa_T k^2)$.

In the model with $\kappa_s = 0$, s is taken as $\pm\frac{1}{2}\Delta S$ in adjacent fingers. The buoyancy-layer scale, now defined by $L = (\mu\kappa_T/g\alpha\overline{T}_z)^{\frac{1}{2}}$, is used to make x dimensionless and L^2/κ_T is the timescale. Setting $w = \kappa_T W/L$, $T = L\overline{T}_z\theta$, leads to the equations

$$\sigma^{-1}W_t - \theta + W = -Q, \tag{17a}$$

$$\theta_t + W - \theta_{xx} = 0, \tag{17b}$$

where Q is $\beta\Delta S/(2\alpha\overline{T}_z L)$ in ascending fingers and changes sign in descending fingers. As $t \to \infty$ the solution tends to that of the steady equations

$$W = \theta - Q, \quad \theta_{xx} = W. \tag{18a, b}$$

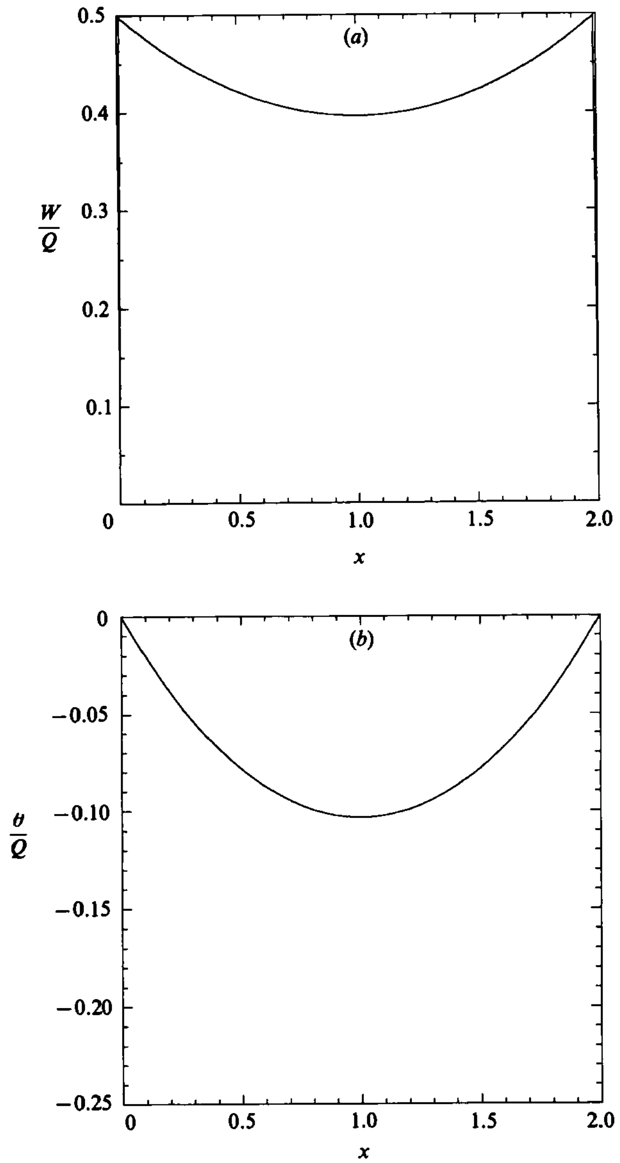

FIGURE 5. (a) W/Q and (b) θ/Q *vs.* x for an ascending finger in a Hele-Shaw cell. The finger width is twice the buoyancy-layer thickness.

The solution is odd and periodic and on the interval $0 < x < a$ it takes the form

$$W = Q\frac{\cosh(x-\tfrac{1}{2}a)}{\cosh\tfrac{1}{2}a}, \tag{19a}$$

$$\theta = Q\left[\frac{\cosh(x-\tfrac{1}{2}a)}{\cosh\tfrac{1}{2}a}-1\right]. \tag{19b}$$

The temperature field is continuous and continuously differentiable, but the Hele-Shaw idealization admits a discontinuity in W at the interface between fingers where $|W|$ achieves its maximum value of Q. Horizontal profiles of W/Q and θ/Q are

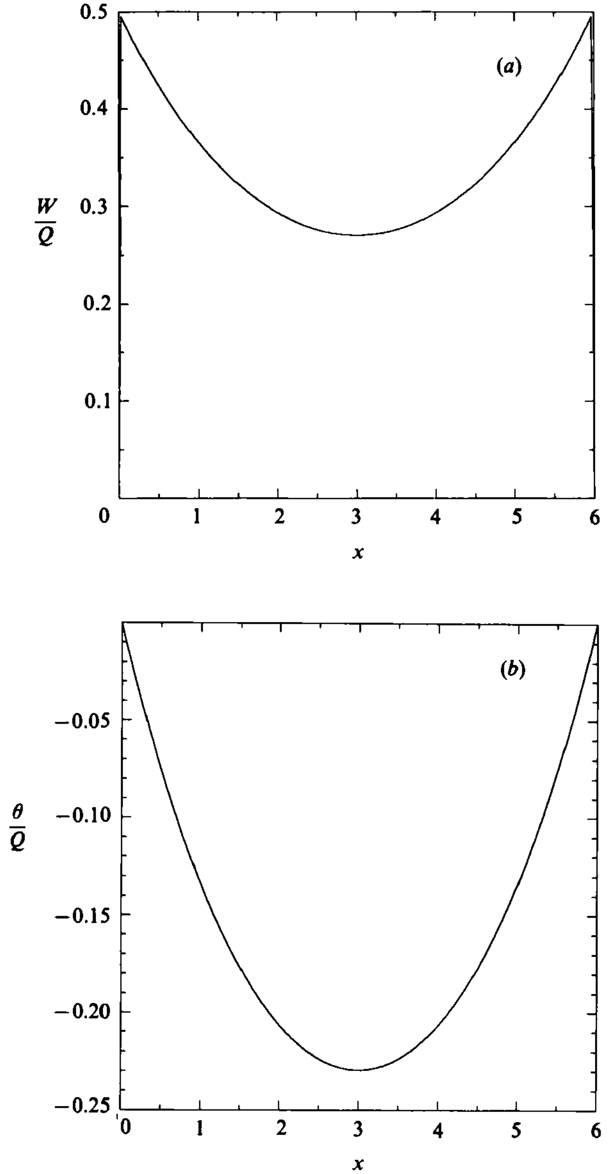

FIGURE 6. (a) W/Q and (b) θ/Q *vs.* x for an ascending finger in a Hele-Shaw cell. The finger width is six times the buoyancy-layer thickness.

shown in figures 5 and 6 for fingers with widths of two and six buoyancy layers respectively.

Heat, salt and buoyancy fluxes are calculated as before. Salt flux achieves its maximum for infinitesimally thin fingers because the horizontally averaged vertical velocity is at its maximum there. Heat flux goes to zero for very thin fingers. Therefore, the buoyancy flux also achieves its maximum in this limit. Figure 7 shows the buoyancy flux and the flux ratio as (monotonic) functions of the finger width.

(A boundary layer with the scale of the gap width would smooth out the

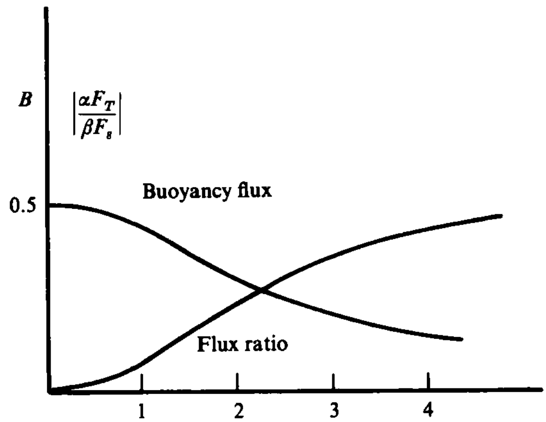

FIGURE 7. Buoyancy flux and flux ratio of heat to salt as functions of finger width in a Hele-Shaw cell. Units on the abscissa correspond to buoyancy-layer thicknesses.

discontinuity in vertical velocity. The salt-diffusion correction of the next section would have the same effect, if the corrected salinity were used as the driving term.)

There are no laboratory measurements of the fluxes in a Hele-Shaw experiment. The physical arrangement is not suitable for a heat–salt experiment starting with two layers because of the heat losses to the walls of the Hele-Shaw cell. However, an experiment with two solutes is feasible. Taylor & Veronis (1985) have reported preliminary results for a sugar–salt experiment but the ratio of the diffusivities is about 1/3 which is rather large for the present theory ($\tau \rightarrow 0$) to apply. Experimentally usable solutes with an appropriately small value of τ are available.

2.3. *Vertical variation of finger width*

As salt fingers penetrate into the reservoirs above and below the interface, the stabilizing mean temperature gradient near the interface decreases with time. The present theory is based on the assumption that the dynamics of the fingers can be analysed by steady-state balances using a local (in time) value of $\overline{T}_z$. However, even with that assumption, fingers of finite length will be associated with a stabilizing temperature gradient that is larger near mid-depth than it is near the reservoirs.

A qualitative description of the structure of the fingers can be realized by allowing $\overline{T}_z$ to vary parametrically with z in the solutions that have been derived for w and T. If $\overline{T}_z$ has a maximum value at $z = 0$ and tends to a constant value at large (compared to the cell width) distance (figure 8*a*) then the buoyancy-layer width will be small at $z = 0$ and large at large $|z|$. Accordingly, the vertical flow in an array of equally spaced fingers will occur across the entire cell at large $|z|$ but near $z = 0$ it will be confined to (buoyancy) boundary layers near the vertical interfaces between cells. That means that there will be regions near the middle of each finger (shown shaded in figure 8*b*) with very little or no vertical motion.

The constant value of $\overline{T}_z$ in the reservoirs at large $|z|$ in the foregoing description determines the broad cell width. In an experiment with two homogeneous reservoirs $\overline{T}_z$ vanishes at large $|z|$, but the width of evolving fingers seems to be controlled by the value of $\overline{T}_z$ near the top and bottom edges of the finger zone, which is what the asymptotically constant $\overline{T}_z$ is meant to represent.

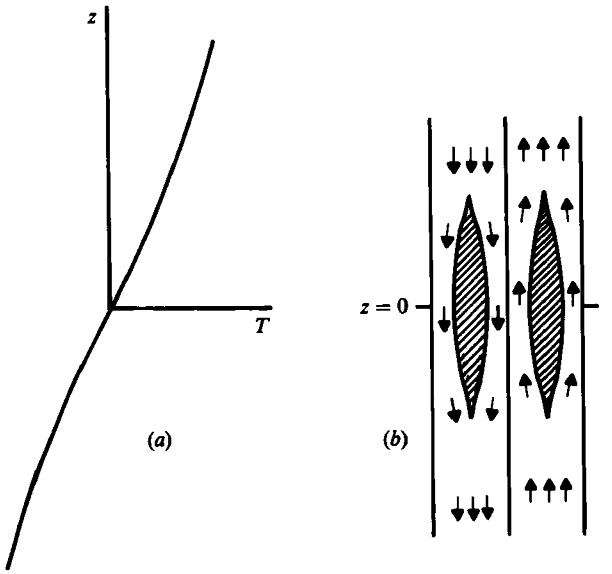

FIGURE 8. (*a*) Mean temperature *vs.* z for a system with maximum $\bar{T}_z$ at $z = 0$ and asymptotically constant $\bar{T}_z$ at large (z). (*b*) A schematic picture of flow in ascending and descending fingers. The large value of $\bar{T}_z$ near $z = 0$ gives rise to thinner buoyancy layers to which the vertical flow is confined. The fluid is essentially quiescent in the shaded regions.

3. Correction for salt diffusion

3.1. *Viscous fluid*

The model solution given above can be expected to give a reasonable description of the flow in a finger zone connecting two reservoirs when the actual finite length of the zone is large compared to the fingers but not so large that horizontal diffusion of salt upsets the hypothesis of constant salinity in each finger. Of course, horizontal salt diffusion will develop some kind of a boundary layer along the interfaces between ascending and descending fingers, but assuming that the latter are not too long, this boundary layer (for small τ) should still be thin compared to the buoyancy layer.

Near the interface between adjacent fingers the vertical velocity varies linearly with x (figure 1). If the vertical coordinate is scaled by h and the horizontal coordinate by H, then with

$$H^3 = \frac{\cos b\pi + \cosh b\pi}{\sin b\pi + \sinh b\pi}\tau\frac{\alpha\bar{T}_z h}{\beta\Delta S}\left(\frac{4\nu\kappa_T}{g\alpha\bar{T}_z}\right)^{\frac{3}{4}} \tag{20}$$

the salinity-boundary-layer equation is free of parameters and becomes

$$xS_z = S_{xx} \tag{21}$$

for the region $0 < z < 1$ and $-\infty < x < \infty$. Both S and S_x are continuous on $0 < z < 1$ as $x \to 0$ from right and left. Furthermore,

$$\left.\begin{aligned} S &= \tfrac{1}{2} \quad \text{at } z = 1 \quad \text{for } x < 0 \quad \text{and for } 0 < z < 1 \quad \text{as} \quad x \to -\infty, \\ S &= -\tfrac{1}{2} \quad \text{at } z = 0 \quad \text{for } x > 0 \quad \text{and for } 0 < z < 1 \quad \text{as} \quad x \to \infty. \end{aligned}\right\} \tag{22}$$

These somewhat unusual conditions are seen to be appropriate both by consideration of the physics of the salinity boundary layer, and by noticing the mathematical structure: the characteristics of the parabolic equation (21) are directed upward for $x > 0$, but downward for $x < 0$.

To solve this problem we consider first a simpler version of it: the same equation (21) but in $x > 0$, $z > 0$, with $S = 0$ on $z = 0$ and $S = f(z)$ on $x = 0$. Invariance of the equation under a scaling transformation leaving x^3/z fixed suggests the existence of similarity solutions depending on a power of this variable. The latter are readily found and the one that satisfies the condition $S = 0$ on $z = 0$ for $x > 0$ is

$$S_0(x, z) = \int_{x/z^{\frac{1}{3}}}^{\infty} \exp\left(-\tfrac{1}{9}\zeta^3\right) \mathrm{d}\zeta. \tag{23}$$

The value of S_0 on $x = 0$ for $z > 0$ is the constant $\Gamma(\frac{1}{3})/3^{\frac{2}{3}}$. Thus S_0, or a multiple of it, solves the problem in the special case of a constant for $f(z)$. The case of a general $f(z)$ can be treated by taking the Laplace transform in z and using the above special solution to help in the evaluation of the inverse transform. The result is

$$S(x, z) = \frac{3^{\frac{2}{3}}}{\Gamma(\frac{1}{3})} \int_{x/z^{\frac{1}{3}}}^{\infty} \exp\left(-\tfrac{1}{9}\zeta^3\right) f\left(z - \frac{x^3}{\zeta^3}\right) \mathrm{d}\zeta, \tag{24}$$

which can readily be checked directly.

By similar methods the same problem with $S = f(z)$ replaced by $S_x = g(z)$ can be solved; from this the relation between f and g corresponding to the same solution S can be determined. It is

$$f(z) = -\frac{\Gamma(\frac{1}{3})}{3^{\frac{1}{3}}\Gamma(\frac{2}{3})} I^{\frac{1}{3}}\{g\}, \tag{25}$$

where I^n corresponds to the Riemann–Liouville fractional integral

$$I^n\{g\} = \frac{1}{\Gamma(n)} \int_0^z (z-\zeta)^{n-1} g(\zeta)\, \mathrm{d}\zeta.$$

We now return to our original salinity-boundary-layer problem. Let $f(z) = S(0, z)$ and $g(z) = S_x(0, z)$, $0 < z < 1$. These are related by (25) but are otherwise unknown. However, they must be approached from both positive and negative x. The function $S + \frac{1}{2}$ vanishes on $z = 0$ and is $f(z) + \frac{1}{2}$ on $x = 0$, so it can be related to $g(z)$ by (25). Similarly, $S(-x, 1-z) - \frac{1}{2}$ vanishes on $z = 1$, is $f(1-z) - \frac{1}{2}$ on $x = 0$, and has x-derivative $-g(1-z)$ on $x = 0$. A second use of (25) provides another relation between f and g, and elimination of f between these two gives the following integral equation for $g(z)$:

$$\int_0^1 |z-\zeta|^{-\frac{2}{3}} g(\zeta)\, \mathrm{d}\zeta = -3^{\frac{1}{3}}\Gamma(\tfrac{2}{3}). \tag{26}$$

Absorbing the constant into g (now written as G) yields.

$$\int_0^1 |z-\zeta|^{-\frac{2}{3}} G(\zeta)\, \mathrm{d}\zeta = 1, \tag{27}$$

the solution of which (see Appendix B) is

$$G(z) = \frac{[z(1-z)]^{-\frac{1}{6}}}{2\pi}. \tag{28}$$

The more general equation with $F(z)$ on the right-hand side of (27) and any exponent between -1 and 0 has been solved by L. N. Howard (1985, unpublished manuscript) in a related way, which gives a representation of the solution more convenient for

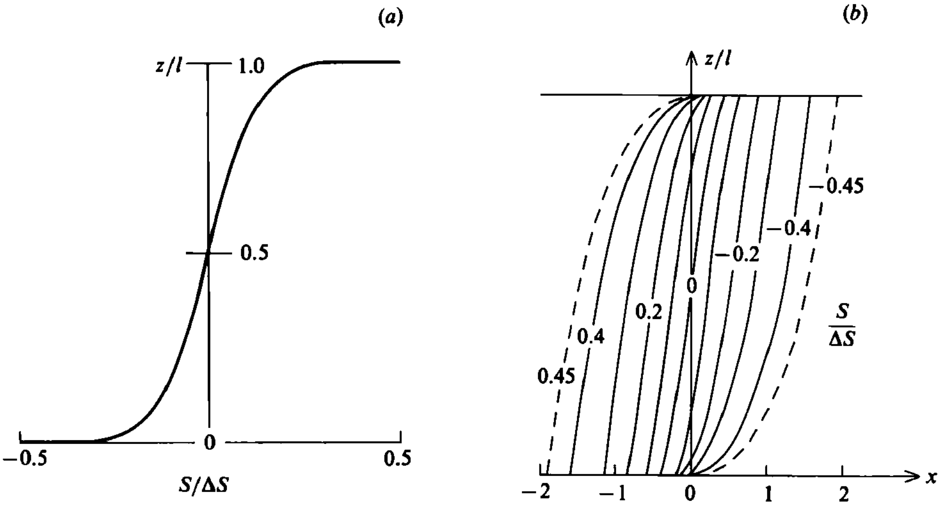

FIGURE 9. (*a*) Salinity correction due to salt diffusion plotted as function of z at $x = 0$, the boundary between fingers. (*b*) Contours of the salinity correction in the boundary layers surrounding the boundary between an ascending finger on the right and a descending finger on the left. Units of x in terms of H in equation (20).

specific cases than the representation by complex integrals presented by Carleman (1922).

From (28) and the preceding formulae the whole structure of the salinity boundary layer can be determined. The simplest procedure is to obtain $f(z)$ from (25) and to integrate (21) numerically using the boundary conditions (22) for each half-strip.

Figure 9 (*a*) exhibits the diffusive correction for $S(z)$ at the interface between rising fluid on the right and descending fluid on the left. The distribution of salinity in the boundary layers surrounding the interface at $x = 0$ is shown in figure 9 (*b*). Near the top of the zone, salt diffusion from the descending cell shifts the $S = 0$ contour to the right; the corresponding effect of fresh water near the bottom shifts it to the left. The overall result is that diffusion broadens the fingers at the incoming level and tapers them to a smaller width as they move through the finger zone.

The diffusive correction has been treated here as a boundary-layer correction with $S = \pm\frac{1}{2}$ at large distance from the interface. Thus, the analysis is applicable provided that the correction penetrates less than half the width of a finger. Since the x-coordinate has been scaled by H in (20), i.e. $x_{\text{dim}} = Hx$, horizontal penetration is appropriately restricted if $Hx < \frac{1}{2}L_m$, from (11). Therefore

$$x < \frac{\sin 2 + \sinh 2^{\frac{1}{3}}}{\cos 2 + \cosh 2}\,\frac{0.85}{(\tau R_\rho)^{\frac{1}{3}}} \approx \frac{0.94}{(\tau R_\rho)^{\frac{1}{3}}}, \tag{29}$$

where $R_\rho = \alpha \overline{T}_z h/\beta\Delta S$. As expected, (29) is less restrictive (the boundary layer is thin) when the system is strongly forced (small R_ρ) or when salt diffusion is weak (small τ).

In the distribution shown in figure 9 (*b*) deepest penetration of the correction occurs at the top right and bottom left. The salinity is within 1 % of the asymptotic values of $\pm\frac{1}{2}$ for $|x| = 3.0$. Therefore, the analysis is applicable for $1 < R_\rho < (32\tau)^{-1}$. Values of R_ρ for typical laboratory heat–salt experiments fall into this range, as does the ratio of heat and salt differences in natural oceanic conditions. However, for salt–sugar experiments $(32\tau)^{-1} < 1$ and the criterion is not satisfied in the regions of greatest penetration of the correction, though it is approximately satisfied at half-depth.

3.2. *Hele-Shaw experiments*

The analysis parallels that in the previous section though there are differences in detail. The dimensional vertical velocity has a constant value, $\pm g\beta\Delta S/2\mu$, on the two sides of the interface and the horizontal coordinate is scaled by $H = (2h\kappa_S \mu/g\beta\Delta S)^{\frac{1}{2}}$. The salt diffusion equation takes the simpler form, $S_z = S_{xx}$, in each half-strip with the same boundary conditions as before. The analysis for the sub-problems is easily carried out with the aid of a Laplace transform in z. The equations relating f to g are

$$f(z) = -\frac{1}{\pi}\int_0^z \frac{g(\zeta)}{(z-\zeta)^{\frac{1}{2}}}\,\mathrm{d}z, \quad x > 0,$$

$$f(1-z) = -\frac{1}{\pi}\int_0^{1-z} \frac{g(\zeta)}{(1-z-\zeta)^{\frac{1}{2}}}\,\mathrm{d}\zeta, \quad x < 0,$$

leading to the final equation

$$\int_0^1 \frac{g(\zeta)/\mathrm{d}\zeta}{|z-\zeta|^{\frac{1}{2}}} = -\pi, \tag{30}$$

the solution of which is

$$g(z) = \frac{-[z(1-z)]^{-\frac{1}{4}}}{(2\pi)^{\frac{1}{2}}}. \tag{31}$$

The vertical distribution of S at $x = 0$ and contours of $S(x, z)$ are qualitatively similar to those of figure 9.

With the present scaling and a finger width of two buoyancy layers, the theory is applicable for

$$x < (1.0\tau R_\rho)^{-\frac{1}{2}}. \tag{32}$$

Since the asymptotic value of S is achieved at about $x = 3.7$, the restriction on R_ρ is $1 < R_\rho < (14\tau)^{-1}$. More feasible Hele-Shaw experiments are with two solutes (no heat) and the ratio of the diffusivities must be considerably smaller than that of sugar to salt (1/3). Easily available solutes, e.g. salt and starch, satisfy the restriction.

4. Discussion

The present model may be helpful in describing the quasi-steady state of salt fingers that have evolved from an initially two-layer configuration, as might be expected to occur at an interface of an intruding water mass with T and S properties different from those of the ambient water. Whether in the laboratory or in nature, the evolution takes place via fingers that both lengthen and broaden and during this period the salinity anomaly is essentially undiminished from one end of the finger to the other. That is observed to happen in experiments with $R_\rho > 1$, i.e. a relatively large salinity difference (Turner 1973, figure 8.8), and in the numerical simulation of salt fingers by Piascek & Toomre (1980). Our analysis is based on the assumption that the fingers are long and essentially steady.

The width of the fingers is controlled by the buoyancy layer, which serves to redistribute the stabilizing temperature field so that the potential energy of the salt field can be released most efficiently. Stern (1960) came to the same conclusion in selecting the optimal wavelength for infinitely longer fingers. However, in our analysis the *structure* of the temperature and velocity fields is also determined by the buoyancy layer. The first-order correction to the salinity field is directly related to the finite length of the fingers.

If fingers of a given width are sufficiently long, the diffusive corrections from the two sides of a finger will come together and salt diffusion will be important across the entire finger. In that case Stern's sinusoidal solution should be applicable over a depth range centred around mid-depth. However, the width of the fingers and the buoyancy flux will be determined by a modified form of the present solution which will be valid near the ends of the finger zone. Thus, the present solution should resolve the indeterminacy of the sinusoidal finger solution.

Another possibility is that fingers will become wider as they lengthen. That is what one would expect of fingers that evolve from a two-layer configuration since the mean temperature gradient diminishes during the evolution. If this dynamic increase of horizontal scales exceeds the lateral penetration of the diffusive correction, there will always be a portion near the middle of the finger in which the salinity anomaly is undiminished, and the present solution will be valid over the entire length of the finger.

We have assumed that a steady model can be used to determine the dynamical balances even though the system that we envisage is globally transient. That assumption may not be valid. There is certainly a transient period near the beginning of a two-layer experiment and we do not know how long the fingers must be before a quasi-steady treatment is applicable. The conclusion that we drew from the analysis of the salt–sugar Hele-Shaw experiment is that the diffusive boundary layers merge. Yet in experiments with large reservoir values of ΔS and ΔT, exiting fingers transport sugar anomalies that appear to be undiminished. The quasi-steady assumption may not be applicable for that case.

We have ignored the transition region between the finger zone and the reservoir where the temperature gradient is smaller than that of the finger zone and would induce wider fingers. We do not know whether that feature can be included without incorporating time dependence.

We have also ignored the effect of vertical salt diffusion where the fingers leave the finger zone and encounter a salinity anomaly of the opposite sign. Although that probably involves only a boundary-layer correction for the exiting finger, it may affect the incoming fingers on either side and alter the conditions where those fingers originate. The same phenomenon at the other end of the finger zone implies that all fingers are affected in the region of origin. We need more information about the conditions that prevail at the boundaries of the finger zone. Because of the small scale in question, help from laboratory experiments would require measurements much more precise than any that have been made up to the present time. Computer simulations may help but they, too, will require a scale resolution that makes the problem difficult, at best.

Our hope is that once we have understood the detailed behaviour near the boundaries of the finger zone, the significant effects in those regions can be approximated so that a more comprehensive model for the evolution of the finger zone can be developed. We may then be in a position to deal with the important problem of how fingers contribute to the formation of mixed layers (reservoirs) above and below.

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Appendix A. Considerations leading to the selection of a horizontal scale

In Stern's idealization, one argument for choosing the horizontal scale is to require the solution to be time independent. However, in that case some further consideration is also needed to fix the amplitude and in fact the other solutions which grow exponentially would seem to deserve further notice.

The determination of horizontal scale in connection with Stern's idealization – or the present one – is an issue somewhat different from that in some stability problems. In the latter, the linearization predicts exponential growth, but this is known from theory or observation to be ultimately limited by nonlinear effects; in such cases the wavelength of maximum growth rate is often (very plausibly) suggested as a good estimate of the horizontal scale that should be expected to occur in an experiment. However, these idealized salt-finger models are not linear stability problems, but families of exact solutions. In Stern's idealization, the exponentially growing solutions are not limited by nonlinearity, and in ours there are no exponentially growing solutions at all.

The determination of the horizontal scale (if it is indeed determined) comes only when the idealization is used as a part of some more complete picture. For instance, in the rather attractive 'interface model' of Schmitt (1979*a*) the scale is determined by supposing the salt-finger zone to have evolved from an initial state in which uniform temperature and salinity gradients exist in a finite layer between two deep homogeneous layers. This layer is supposed to become filled with salt fingers, described by Stern's idealization (as if the gradient layer were infinite in vertical extent) using the scale that gives maximum growth rate for the initial gradients; the scale is then supposed to remain fixed throughout the evolution.

The evolution process is not described in detail, but Schmitt argues that it can be expected to reduce the mean salt gradient inside the layer as the amplitude of the salt fingers increases (with fairly rapid changes in salinity appearing at the edges of the layer), and come to an end when $\bar{S}_z$ has been sufficiently reduced that the original horizontal scale corresponds to the steady solutions of Stern's idealization. This reasoning is rather similar to the description of a linear instability ultimately limited by nonlinearity, but the circumstances are not really the same. Here, the limiting is a consequence of the finite thickness, which makes possible the reduction of the internal mean salt gradient. That is not a part of Stern's idealized salt-finger model, but Schmitt's idea of using Stern's model as a description at the start of the evolution process seems entirely appropriate and the selection of the scale of maximum growth rate seems very reasonable. Initially, the vertical velocity is small and horizontal diffusion could generate the sinusoidal profile a relatively short distance into the finger.

He also uses Stern's model as a description of the final state; there is a minor difficulty in doing so, for without a detailed description of the evolution it is not entirely clear which amplitude of the steady solution should be expected in the final state. Schmitt's suggestion – also physically plausible – is that the amplitude should be such that the extremal salinity in the finger is about the same as that of the homogeneous region from which it comes. But in the final state it is at least conceivable that the vertical velocity may be so large that very little horizontal diffusion can occur before a fluid parcel has completely traversed the salt-finger layer. To assess the situation we give the following argument.

Suppose that the layer of initially uniform gradients $\bar{T}_z$, $\bar{S}_z$, is of thickness h, and that the salinity difference across it is ΔS. The horizontal wavenumber k is selected, as suggested by Schmitt, to be that corresponding to the maximum growth rate for the initial gradient. After the evolution the mean salinity gradient has been reduced to a value $\bar{S}'_z$, which makes k correspond to a steady solution, and from (3a) is given by

$$\bar{S}'_z = \tau\left(\frac{k^4\nu\kappa_T}{g+\alpha\bar{T}_z}\right)\Big/\beta. \tag{A 1}$$

Using this and the equations of Stern's idealization in the steady case ((2), with $\lambda = 0$) one readily finds

$$w = -\frac{k^2\kappa_T\beta}{k^4\nu\kappa_T/g+\alpha\bar{T}_z}s. \tag{A 2}$$

We now set $s = \frac{1}{2}\Delta S$, and on introducing the dimensionless breadth $b = 1/(kL)$ we get

$$|w| = \frac{\kappa_T}{L^2}\frac{\beta\Delta S}{\alpha\bar{T}_z}\frac{2b^2}{1+4b^4}. \tag{A 3}$$

Now Stern's idealization can be regarded as appropriate in this final state provided that the residence time $h/|w|$ in a finger is long enough that the corresponding horizontal diffusion scale $\pi(\kappa_S h/|w|)^{\frac{1}{2}}$ is large compared with the finger breadth πbL, i.e. provided $h \gg b^2L^2|w|/\kappa_S$. Using the above formula for $|w|$ this condition becomes

$$h \gg \frac{1}{\tau}\frac{\beta\Delta S}{\alpha\bar{T}_z}\frac{2b^4}{1+4b^4}, \tag{A 4}$$

which, introducing the density ratio $R_\rho = \alpha h\bar{T}_z/\beta\Delta S$, can be written

$$\tau R_\rho \gg \frac{2b^4}{1+4b^4}. \tag{A 5}$$

For salt-finger problems R_ρ must lie in the range 1–τ^{-1}, so the left side is at most 1, and for small τ it is considerably less than 1 unless the salt driving is very weak. To estimate the right-hand side we need to know the value of b corresponding to the maximum growth rate for the initial state. Schmitt (1979a) has given values of a dimensionless wavenumber M, which in our notation is equal to $1/(b\surd 2)$, for the cases $\sigma = 7$, $\tau = \frac{1}{100}$ $(1 \leqslant R_\rho \leqslant 100)$ and $\sigma = 1000$, $\tau = \frac{1}{3}$ $(1 \leqslant R_\rho \leqslant 3)$.

In both cases $M < 1$ which means that $4b^4 > 1$ and so the right-hand side of (A 5) is at least $\frac{1}{4}$. Thus the condition (A 5) for the appropriateness of Stern's idealization in the final state can be only marginally satisfied in the most favourable case and is not satisfied if there is strong salt driving (R_ρ near 1). It can be shown that

$$4b^4 \geqslant \frac{\sigma+\tau}{\sigma(1-\tau)}[1+5\epsilon-\tfrac{1}{2}\epsilon^2+(4+\epsilon)(\epsilon+\tfrac{1}{4}\epsilon^2)^{\frac{1}{2}}],$$

where the equality holds for maximum growth rate, and $\epsilon = \sigma\tau/(\sigma+\tau)$, from which it is easily seen that $4b^4 \geqslant 1$ for all σ and $\tau < 1$, not just for the values calculated by Schmitt. Indeed, it is possible to show that $\tau R_\rho \leqslant 4b^4/(1+4b^4)$ for all σ, all $\tau \leqslant 1$, and all R_ρ on $(1, 1/\tau)$, so the condition (A 5) is never really satisfied.

When the strong inequality in (A 5) is reversed, diffusion plays little role in the salt fingers; this is exactly the situation for which the idealization presented here is appropriate, and it can be used instead of Stern's idealization to make a model along the lines of Schmitt's for cases of small τ and τR_ρ sufficiently small. The basic physical picture is the same, and the horizontal scale is determined as suggested by Schmitt. But the final state is approximated by the idealization described here, rather than by Stern's. The main quantitative results are very little different from those calculated by Schmitt, because the amplitude of the salt perturbation assumed by him is the same as that of the present model and the profiles are not greatly different unless b turns out to be appreciably larger than 1, which is probably only the case when τR_ρ is near 1. (If it should turn out that b exceeded 2, our model would suggest that the fingers would be split so that the horizontal scale would not in fact be determined by Schmitt's condition – but this seems to be rarely if ever the case.) The main qualitative difference is perhaps the square-wave profile of salt in the finger, together with the fact that the description of the final state can now be justified. It seems to us also that the explicit recognition in the idealization of the finite height of the finger zone, which comes in through the driving by ΔS rather than by $\bar{S}_z$, makes the description more physically satisfactory. Of course the calculations we give here are for $\tau = 0$, and are of doubtful applicability when τ is as large as $\frac{1}{3}$ – the sugar–salt case. The salinity-boundary-layer approach discussed in §3 could be utilized to calculate some correction to the fluxes, etc. for finite τ. However, if τ is as large as $\frac{1}{3}$, then either one should use Stern's idealization (even though it is not really justified!) in the manner of Schmitt, if τR_ρ is near 1, or, if $\tau R_\rho \ll 1$, one should recognize that neither the thermal nor the salinity fields will be much affected by horizontal diffusion, and try to take that into account.

The determination of a horizontal scale by Schmitt's argument may not be appropriate for all fields of salt fingers – in particular the interesting case of evolution from an initial temperature and salinity jump seems rather different – and the finger width in a particular salt-finger zone may depend on how it was actually established. We hope that the present idealized model may also be useful as a part of other more complete theories, which may fix the horizontal scale on some other basis. For given values of ΔS and $\bar{T}_z$ the fluid would release the potential energy available in the salinity field most efficiently by selecting the finger width that leads to a maximum of the buoyancy flux B. This is a plausible criterion for choosing an appropriate scale, and was used, with some other considerations, in the model of Stern (1976) mentioned above, to determine the parameters of a particular steady solution of (2). This criterion could also be used to fix b in the present model.

Appendix B

To evaluate

$$\int_0^1 |x-y|^{-\beta} f(y)\, \mathrm{d}y = 1 \tag{B 1}$$

change the notation by setting

$$\beta = 2\alpha, \quad f(y) = [y(1-y)]^{(\beta-1)/2} F(2y-1), \quad 2y-1 = w, \quad 2x-1 = z \tag{B 2}$$

to obtain

$$\int_{-1}^1 |z-w|^{-2\alpha}(1-w^2)^{\alpha-\frac{1}{2}} F(w)\, \mathrm{d}w = 1, \tag{B 3}$$

where we assume $|\alpha| < \frac{1}{2}$ to ensure the existence of the integral for all continuous F. We now show that when $F(w) = 1$, the integral in (B 3) is a constant C for $|z| \leqslant 1$. Thus $F = 1/C$ solves (B 3). C is evaluated in the following.

Take $F(w) = 1$ and change to the variable u defined by

$$w = \frac{z+u}{1+zu}, \tag{B 4}$$

which makes $w = -1, z, 1$ correspond to $u = -1, 0, 1$ respectively. Then

$$1-w^2 = \frac{(1-z^2)(1-u^2)}{(1+zu)^2}, \quad w-z = \frac{u(1-z^2)}{(1+zu)}, \quad dw = du\frac{(1-z^2)}{(1+zu)^2}$$

and this gives

$$(1-z^2)^{\frac{1}{2}-\alpha}\int_{-1}^{1} \frac{(1-u^2)^{\alpha-\frac{1}{2}}|u|^{-2\alpha}}{(1-zu)}\, du. \tag{B 5}$$

The integral I in (B 5) can be rewritten as

$$\int_0^1 \cdots \frac{du}{1+zu} + \int_0^1 \cdots \frac{du}{1-zu} = 2\int_0^1 \cdots \frac{du}{1-z^2u^2}. \tag{B 6}$$

Using $\zeta = z^2$ and $v = u^2$ as variables leads to

$$I(\zeta) = \int_0^1 (1-v)^{\alpha-\frac{1}{2}} v^{-\alpha-\frac{1}{2}} \frac{dv}{(1-\zeta v)}. \tag{B 7}$$

Then

$$\begin{aligned}
\frac{dI}{d\zeta} &= \int_0^1 (1-v)^{\alpha-\frac{1}{2}} v^{-\alpha+\frac{1}{2}} \frac{dv}{(1-\zeta v)^2} \\
&= \int_0^1 (1-v)^{\alpha-\frac{1}{2}} v^{-\alpha+\frac{1}{2}} \frac{1}{\zeta}\, d\left[\frac{1}{1-v\zeta} - \frac{1}{1-\zeta}\right] \\
&= \int_0^1 (1-v)^{\alpha-\frac{1}{2}} v^{-\alpha+\frac{1}{2}}\, d\left[-\frac{1-v}{(1-\zeta)(1-\zeta v)}\right] \\
&= \int_{-1}^1 \frac{(1-v)\, dv}{(1-\zeta)(1-\zeta v)} (\tfrac{1}{2}-\alpha)\left[(1-v)^{\alpha-\frac{3}{2}} v^{-\alpha+\frac{1}{2}} + (1-v)^{\alpha-\frac{1}{2}} v^{-\alpha-\frac{1}{2}}\right] \\
&= \frac{(\frac{1}{2}-\alpha)\, I(\zeta)}{1-\zeta}.
\end{aligned}$$

Therefore

$$I(\zeta) = C(1-\zeta)^{\alpha-\frac{1}{2}}. \tag{B 8}$$

In view of (B 5) this shows that the integral on the left in (B 3) is indeed a constant when $F = 1$ and $|z| \leqslant 1$. (We have tacitly assumed $|z| \leqslant 1$ in using (B 4); the integral is not constant outside this interval.) Since $I(0)$ is a beta function, the constant C is easily evaluated. For a viscous fluid the value of β in (B 1) is $\frac{2}{3}$ and the constant C in (B 8) is 2π. For a Hele-Shaw cell $\beta = \frac{1}{2}$ and C is $\pi\sqrt{2}$.

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